Adomian Decomposition Method for the Solution of Differential Equation

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ABSTRACT: This paper explores the Adomian Decomposition Method (ADM), a robust technique for solving linear and nonlinear differential equations introduced by G. Adomian. The main goal of this study is to offer an introduction to ADM, which could be beneficial for scientists seeking to grasp the method before delving into more complex applications. ADM addresses differential equations (both linear and nonlinear) by representing the solution as a series, with terms determined sequentially through a recursive relation. The paper presents an overview of ADM applied to standard differential equations, using various examples to illustrate the techniques. The fundamental principles and procedures are explained, and the Adomian solution is compared with the exact solution obtained through traditional mathematical methods to demonstrate their equivalence.

Keywords:

Adomian points, Adomian polynomial, differential equation, Series solutions, Adomian Decomposition Method.

1. INTRODUCTION

The ADM, initially proposed by G. Adomian and widely adopted in the 1980s, is a versatile tool for solving a broad spectrum of linear and non-linear PDEs, with significant applications in science and engineering [1]. One of the main advantages of ADM is its ability to analytically approximate large classes of non-stochastic differential equations without relying on discretization methods, perturbation methods, linearization methods, or closure approximations, which often lead to extensive numerical calculations [2]. Notably, ADM can solve differential and integral equations expressed as series, with terms that can be recursively obtained using Adomian polynomials. A key benefit of the method is the fast convergence of the series solution, which significantly reduces computing time.

The purpose of this paper is to provide a comprehensive pedagogical and educational overview of ADM and its applications. We focus specifically on a variety of common first and second-order differential equations (both linear and nonlinear) and offer a detailed description of the Adomian method for solving them. In every case, we demonstrate the equivalence of the Adomian solution by comparing it to the exact solutions of specific differential equations.

Recently, there has been considerable interest in exploring ADM, as it allows for the analysis of solutions and characteristics of a wide range of first and second-order ordinary differential equations (ODEs). These solutions can address various mathematical problems and model a diverse array of physical processes.

2. LITERATURE REVIEW

Numerous fields have extensively investigated and utilized the ADM. Its efficacy in the solutions of Fokker-Planck equation and nonlinear heat equation is demonstrated by studies [3] and [4], respectively. However, study [5] highlights that ADM's usefulness is limited under certain boundary conditions, raising some doubts about its reliability. Study [3] provides a thorough analysis of ADM and its variations, emphasizing its ability to solve fractional differential equations. Despite some drawbacks, the literature indicates that ADM is a versatile method capable of addressing challenging problems.

Aslano and Abu-Alshaikh [6] examined singular initial value problems in second-order ODEs of the Lane-Emden type. To address the singularity challenges in inhomogeneous, linear, and nonlinear Lane-Emden-like equations, this work presents an advancement of the Adomian decomposition approach. This is particularly useful when the singularity occurs on the right-hand side of such equations. A comprehensive understanding of these initial value problems is provided through numerous examples. When exact solutions are available, the numerical results are compared accordingly.

Mak, Leung, and Harko [7] detailed the application of the Adomian method in solving common ODEs such as Abel, Riccati, and Bernoulli equations, demonstrating the method's effectiveness by comparing its solutions to exact solutions of specific differential equations, establishing their complete equivalence. They also explored second and fifth-order ordinary differential equations. An essential addition to the traditional ADM approach, the Laplace-Adomian decomposition technique, is introduced to solve a second-order nonlinear differential equation. They further apply this method to the second-order Kolmogorov differential equation, which is crucial for explaining various physical processes. Additionally, they discuss three significant applications of ADM in astronomy and astrophysics: solving the Lane-Emden equation, the Kepler equation, and the general relativistic differential equation that describes the motion of massive particles in static Schwarzschild geometry with spherical symmetry.

3. ADOMIAN DECOMPOSITION METHOD

Adomian decomposition schemes are employed to address various problems whose mathematical formulations result in an equation or a system of integral, differential, or integro-differential equations. Consider the nonlinear equation in general form as below:

$$\ell\psi + \Re\psi + \widehat{N}\psi = \widehat{f(s)}. \tag{1}$$

Given that ℓ is assumed to be the highest order derivative, readily invertible, $\Re$ is the less-ordered linear differential operator. $\widehat{N}\psi$ is consider to be the nonlinear terms and $\widehat{f(s)}$ is the source term. Applying the inverse operator ℓ^{-1} on the equation (1), and applying conditions we get

$$\psi(s) = \widehat{g(s)} - \ell^{-1}(\Re\psi) - \ell^{-1}(\widehat{N}\psi), \tag{2}$$

where the terms resulting from using the specified conditions and integrating the source term $\widehat{f(s)}$ are represented by the function $\widehat{g(s)}$. For nonlinear DE, the nonlinear operator $\widehat{N}\psi = F(\psi)$ is represented by an infinite series called Adomian polynomials. This polynomial will only be use if we have a nonlinear term of $\widehat{N}(\psi)$. We decompose the ψ-term of the unknown function in the form of infinite series. Now, let

$$\psi(s) = \sum_{c=0}^{\infty} \psi_c. \tag{3}$$

Also consider the non-linear term in (1) which can be decompose as follow

$$\widehat{N}(\psi) = \sum_{c=0}^{\infty} \rho_c\,, \tag{4}$$

where the $\rho_c's$ are polynomials of $\psi_0, \psi_1, \ldots, \psi_c$ called Adomian's polynomials and are obtained by the formulae

$$\rho_c = \frac{1}{c!}\frac{d^c}{d\lambda^c}[\widehat{N}\left(\sum_{\alpha=0}^{\infty}\lambda^{\alpha}\psi_{\alpha}\right)]_{\lambda=0}, \quad c = 1,2,3,4 \ldots. \tag{5}$$

Putting (3) and (4) in (2) we get

$$\sum_{c=0}^{\infty}\psi_c = g(s) - \ell^{-1}\Re\left[\sum_{c=0}^{\infty}\psi_c\right] - \ell^{-1}\widehat{N}\left[\sum_{c=0}^{\infty}\rho_c\right], \tag{6}$$

$$\sum_{c=-1}^{\infty}\psi_{c+1} = g(s) - \sum_{c=0}^{\infty}[\ \ell^{-1}\Re[\psi_c] - \ell^{-1}\widehat{N}[\rho_c]],$$

$$\psi_0 + \sum_{c=0}^{\infty}\psi_{c+1} = g(s) - \sum_{c=0}^{\infty}[\ \ell^{-1}\Re[\psi_c] - \ell^{-1}\widehat{N}[\rho_c]], \tag{7}$$

NB: $\sum_{c=1}^{\infty}\psi_{c+1}$ helps to get an iteration relation so that we can get the unknown terms for ψ in (3).
Now, from (7), we get

$$\psi_0(s) = g(s), \tag{8}$$

$$\psi_{c+1} = \ell^{-1}\Re[\psi_c] - \ell^{-1}\widehat{N}[\rho_c], \quad c = 1,2,3,4 \ldots. \tag{9}$$

Now, (8) and (9) are iteration scheme to be follow. By taking the finite number of terms from (9) we get an approximate or exact solutions as follow

$$\psi(t) \approx \sum_{c=0}^{N}\psi_c\ ,$$

or,

$$\psi(s) = \lim_{c\to\infty}\sum_{c=0}^{N}\psi_c(s).$$

4. THE ADM FOR FIRST ORDER DIFFERENTIAL EQUATION

In this section, we deal with first order differential equations. Consider the general form of first ODE

$$\frac{d\psi}{ds} + \widehat{K(s)}\psi = \widehat{g(s)}\ , \tag{10}$$

where $\widehat{K(s)}$ and $\widehat{g(s)}$ are any given function of s. (10) Can only be solve together with the initial condition $\psi(0)$ =ψ_0.
Comparing (1) and (10) we have
$\ell[\psi] = \frac{d\psi}{ds}$, $\Re[\psi] = 0$, $\widehat{N}[\psi] = 0$, and $\widehat{f(s)} = \widehat{g(s)}$ and

$$\ell^{-1} = \int_0^s (.)\,.$$

Operating ℓ^{-1} in (10) we get

$$\int_0^s \frac{d\psi}{ds}ds = \int_0^s \widehat{g(s)}\,ds - \int_0^s \widehat{K(s)}\psi\,ds, \tag{11}$$

$$\psi(s) - \psi(0) = \int_0^s \widehat{g(s)}\,ds - \int_0^t \widehat{K(s)}\psi\,ds,$$

$$\psi(s) = \psi(0) + \int_0^s \widehat{g(s)}\, ds - \int_0^s \widehat{K(s)}\psi\, ds\,, \tag{12}$$

Substituting (3) in (12) we obtain

$$\sum_{c=0}^{\infty} \psi_c(s) = y(0) + \int_0^s \widehat{Q(s)}\, ds - \int_0^s \widehat{K(s)} \sum_{c=0}^{\infty} \psi_c\,(s)\, ds$$

$$\psi_0(s) + \sum_{c=0}^{\infty} \psi_{c+1}(s) = \psi(0) + \int_0^s \widehat{Q(s)}\, dp - \int_0^s \widehat{P(s)} \sum_{n=0}^{\infty} \psi_c\,(s)\, dp$$

To determine the components $\psi_\alpha(s)$, we use the recursive relation

$$\psi_0(s) = \psi(0) + \int_0^s \widehat{g(s)}\, ds \tag{13}$$

$$\psi_{\alpha+1}(s) = -\int_0^t \widehat{K(s)}\psi_\alpha(s)\, ds \tag{14}$$

Example

$$\frac{d\psi}{ds} + s\psi = 4s \qquad \psi(0) = 2. \tag{15}$$

The general solution using the condition $\psi(0) = 2$ is given by;

$$\psi = 4 - 2e^{-\frac{2}{s^2}}\,. \tag{16}$$

Now consider (15) we have $\widehat{K(s)} = s$, $\widehat{g(s)} = 4s$. Thus the power series solution can be obtain using the iterative scheme in (13) and (14):

$$\psi_0(s) = \psi(0) + \int_0^s \widehat{g(s)} ds. \tag{17}$$

$$\psi_{\alpha+1}(s) = -\int_0^s \widehat{K(s)}\psi_\alpha(s) ds. \tag{18}$$

Using (15), (17) & (18), we obtain

$$\psi_0(s) = \psi(0) + \int_0^s 4s ds = 2 + 2s^2,$$

$$\psi_1(s) = -\int_0^s \widehat{K(s)}\psi_0(s) ds = -s^2 - \frac{s^4}{2},$$

$$\psi_2(s) = -\int_0^s \widehat{K(s)}\psi_1(s) ds = \frac{s^4}{4} + \frac{s^6}{12},$$

$$\psi_3(s) = -\int_0^s \widehat{K(s)}\psi_2(s) ds = -\frac{s^6}{24} - \frac{s^8}{96},$$

$$\psi_4(s) = -\int_0^s \widehat{K(s)}\psi_3(s) ds = \frac{s^8}{192} + \frac{s^{10}}{960},$$

$$\psi_5(s) = -\int_0^s \widehat{K(s)}\psi_4(s) ds = -\frac{s^{10}}{1920} - \frac{s^{12}}{11520},$$

$$\psi(s) \approx \psi_0(s) + \psi_1(s) + \psi_2(s) + \psi_3(s) + \psi_4(s) + \psi_5(s)\,,$$

$$= 2 + s^2 - \frac{s^4}{4} + \frac{s^6}{24} - \frac{s^8}{192} + \frac{s^{10}}{1920} - \frac{s^{12}}{11520} \ldots. \quad (19)$$

Thus,

$$\psi(s) \approx 2 + s^2 - \frac{s^4}{4} + \frac{s^6}{24} - \frac{s^8}{192} + \frac{s^{10}}{1920} - \frac{s^{12}}{11520} \ldots. \quad (20)$$

Also, by series expansion of general solution in (16) we obtain

$$\psi(s) = 4 - 2e^{-\frac{2}{s^2}} = 2 + s^2 - \frac{s^4}{4} + \frac{s^6}{24} - \frac{s^8}{192} + \frac{s^{10}}{1920} - \frac{s^{12}}{11520} \ldots. \quad (21)$$

Apparently, the solution (20) obtained by ADM is same as the (21) exact solution.

5. SOLUTION OF NON-LINEAR FIRST ORDER DIFFERENTIAL EQUATION USING ADM

The general method for solving nonlinear differential equations was explained at the beginning of this paper. In this section, we will work through some examples to further illustrate the Adomian Techniques. Consider a differential equation of the form as given in the following example.

Example

$$\frac{d\psi}{ds} - \psi^2 = 0, \qquad \psi(0) = 1. \quad (22)$$

Here, the highest order derivative is $\frac{d\psi}{ds} = \ell(\psi)$, $\ell^{-1} = \int_0^s (.)\, ds$ and non-linear term $\widehat{N}(\psi) = \psi^2$.

operating ℓ^{-1} on (22) we get,

$$\int_0^s \frac{d\psi}{ds} ds - \ell^{-1}(\psi^2) = 0\,. \quad (23)$$

$$\psi(s) - y(0) - \ell^{-1}(\psi^2) = 0.$$

$$\psi(s) - 1 - \ell^{-1}(\psi^2) = 0. \quad (24)$$

$$\widehat{N}(\psi) = \sum_{c=0}^{\infty} \rho_c\,, \quad (25)$$

where,

$$\rho_c = \frac{1}{c!}\frac{d^c}{d\lambda^c}\widehat{N}[\sum_{\alpha=0}^{N} \psi_\alpha \lambda^\alpha]_{\lambda=0}.$$

Let the solution of (22) be in the form

$$\psi = \sum_{c=0}^{\infty} \psi_c\,. \quad (26)$$

Substituting (25) & (26) in (24) we get

$$\sum_{c=0}^{\infty} \psi_c = 1 + \ell^{-1}\left[\sum_{c=0}^{\infty} \rho_c\right],$$

$$\psi_0 + \sum_{c=0}^{\infty} \psi_{c+1} = 1 + \sum_{c=0}^{\infty} \ell^{-1}(\rho_0), \quad (27)$$

Clearly, from (27) we have

$$\psi_0 = 1.$$

$$\psi_{c+1} = \ell^{-1}(\rho_c) \qquad c = 0,1,2,3, \ldots. \quad (28)$$

Now, we can use (5) to obtain the $\rho_c{}'s$

$$\rho_0 = \frac{1}{0!}\widehat{N}[\psi_0].$$
$$\rho_0 = \widehat{N}[\psi_0] = {\psi_0}^2.$$
$$\rho_1 = \frac{1}{1!}\frac{d}{d\lambda}\widehat{N}[\sum_{\alpha=0}^{1}\psi_\alpha\lambda^\alpha]_{\lambda=0}.$$
$$\rho_1 = 2\psi_0\psi_1.$$
$$\rho_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}N[\sum_{\alpha=0}^{2}\psi_\alpha\lambda^\alpha]_{\lambda=0}.$$
$$\rho_2 = 2\psi_0\psi_2 + {\psi_1}^2.$$
$$\vdots$$

Now, we will use the iterative scheme in (28) to get the values of ${\psi_c}^s$.

$$\psi_1 = \ell^{-1}[\rho_0] = \int_0^s [\,1]ds = s.$$
$$\psi_2 = \ell^{-1}[\rho_1] = \int_0^s [2\psi_0\psi_1] = s^2.$$
$$\psi_3 = \ell^{-1}[\rho_2] = \int_0^s \left[2\psi_0\psi_2 + {\psi_1}^2\right] = s^3.$$
$$\psi_4 = \ell^{-1}[\rho_3] = \int_0^s [2\psi_0\psi_3 + 2\psi_1\psi_2] = s^4.$$
$$\vdots$$

Using the recursive relation of (27) and iterative scheme of (28) we have

$$\psi_1 = s.$$
$$\psi_2 = s^2.$$
$$\psi_3 = s^3.$$
$$\psi_4 = s^4.$$
$$\vdots$$

Putting these results above in (26) we obtain

$$\psi = \psi_0 + \psi_1 + \psi_2 + \psi_3 + \psi_4 + \cdots.$$
$$\psi = 1 + s + s^2 + s^3 + s^4 + \cdots.$$

Since this is an infinite geometric series can be sum as

$$\psi = \frac{1}{1-s}\,. \tag{29}$$

And we know that this series converges if $|s| < 1$.

Now, we compare the solution of the given equation using ADM and solution using the conventional method. Consider (22)

$$\frac{d\psi}{ds} - \psi^2 = 0 \qquad \psi(0) = 1.$$
$$\frac{d\psi}{\psi^2}d\psi = ds.$$

$$\int \frac{d\psi}{\psi^2} d\psi = \int ds + \alpha.$$

$$\frac{-1}{\psi} = s + \alpha. \tag{30}$$

Using the condition,

$$\psi(0) = 1.$$

We get

$$\alpha = -1.$$

Putting the value of α in (30) we obtain the general solution as

$$\psi = \frac{1}{1 - s}. \tag{31}$$

Observing (29) and (31) the ADM arrived at the exact solution of the given nonlinear ODE. It's important to note that, ADM is known by its virtue in obtaining the approximate solution of mathematical equation. However, in this case it is obvious to see that the ADM yield exact solution. As explained above the G. Adomian method may yield exact solution in some instance but not often.

6. CONCLUSION

We have briefly reviewed some of the key features of the powerful approach to solving linear and nonlinear DEs introduced by G. Adomian, known as the ADM. This method decomposes the solution to a nonlinear operator equation into a set of analytic functions. After presenting the fundamentals of the method, we extensively discussed ADM for standard DEs, including linear and nonlinear ordinary DEs, using various examples to clarify the techniques. Each example provided a thorough explanation of the basic formalism and specific procedure, demonstrating that the Adomian solution precisely matches the exact analytic solution obtained through conventional mathematical techniques.

The primary aim of this study is to offer an introduction to ADM that could be valuable for scientists seeking to understand this method before progressing to more complex applications. Hopefully, this brief introduction will inspire scientists from various fields to explore this fascinating and productive area of study further, particularly for its effectiveness in handling complex mathematical models that describe natural phenomena.

Future researchers are encouraged to focus on convergence analysis, as rigorous proofs of convergence are still lacking in some cases, especially for complex or modified ADM variants. Additionally, the development of methods for error estimation and control is necessary to ensure accurate and reliable solutions by effectively managing and quantifying errors.

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